Multivariate calculus reminders

Multivariate functions

We let $f : \mathbb{R}^p \to \mathbb{R}.$

Gradient: $\nabla f(x) \in \mathbb{R}^p, \quad [\nabla f(x)]_i = \dfrac{\partial f}{\partial x_i}(x)$

If it exists, f is **differentiable**

Hessian : $\nabla^2 f(x) \in \mathbb{R}^{p \times p}, \quad [\nabla^2 f(x)]_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}(x)$

Taylor expansion

We let $f : \mathbb{R}^p \to \mathbb{R}, \;\; x, \;\; \varepsilon \in \mathbb{R}^p$

$$f(x + \varepsilon) = f(x) + \langle \nabla f(x), \varepsilon \rangle + \frac{1}{2} \langle \varepsilon, \nabla^2 f(x) \varepsilon \rangle + o(\|\varepsilon\|^2)$$

f locally looks like a quadratic function

Taylor expansion

Question: let $A \in \mathbb{R}^{n \times p}$ and define

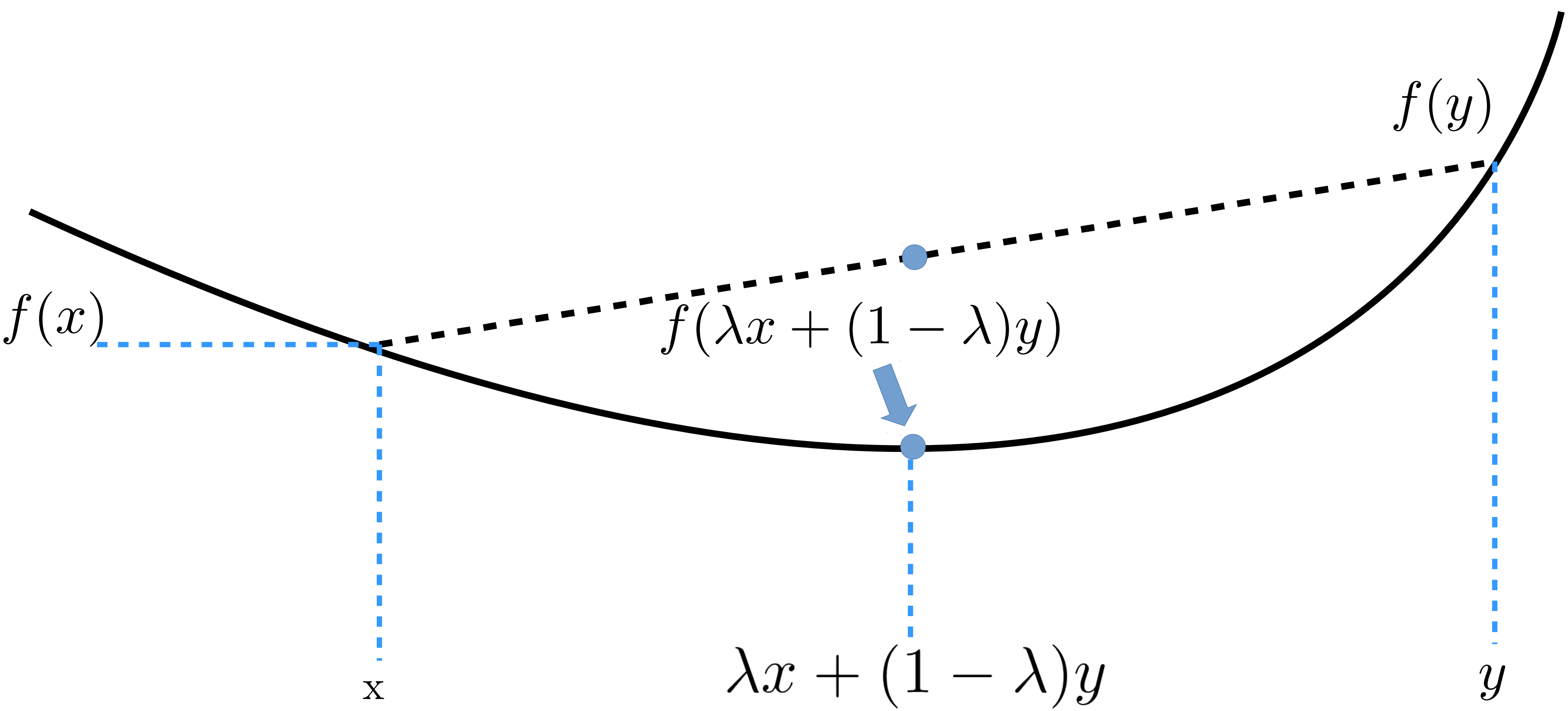
$$f(x) = \frac{1}{2}\|Ax\|^2$$

What are the gradient / Hessian ?

Convexity, zero-th order definition

We let $f : \mathbb{R}^p \to \mathbb{R}$. f is convex if

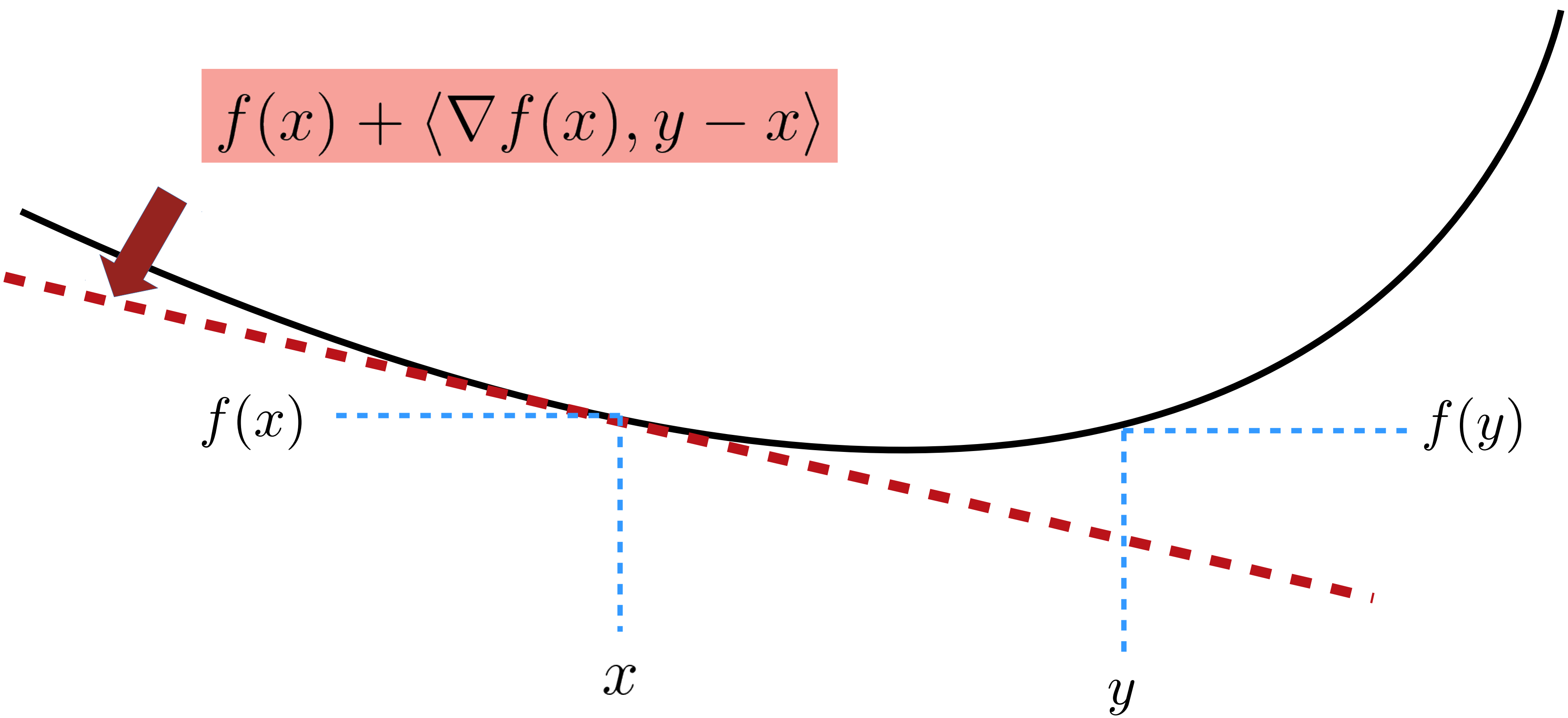
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in \mathbb{R}^p, \lambda \in [0, 1]$$



Convexity, first order definition

We let $f : \mathbb{R}^p \to \mathbb{R}$. f is convex if

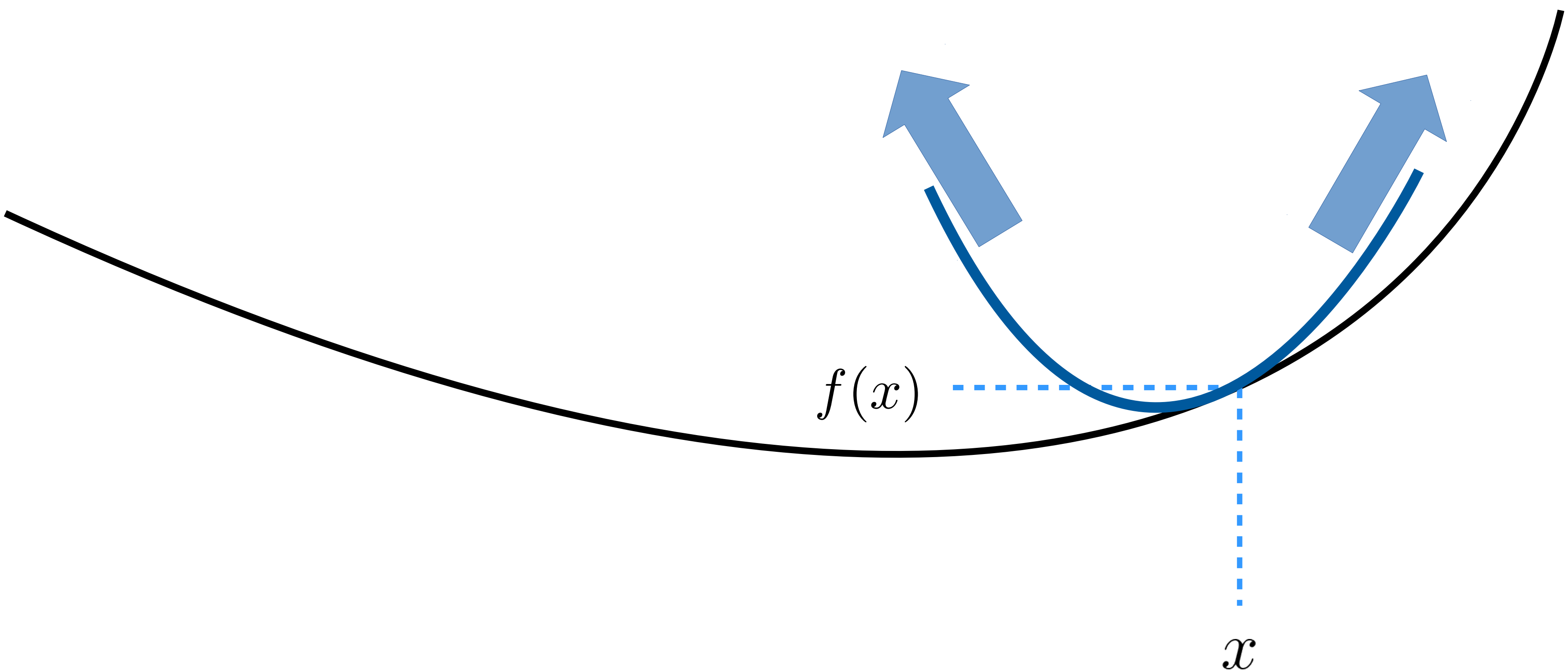
$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \quad \forall x, y \in \mathbb{R}^p$$



Convexity, second order definition

We let $f : \mathbb{R}^p \to \mathbb{R}$. f is convex if

$$\nabla^2 f(x) \succeq 0, \quad \forall x \in \mathbb{R}^p$$



Questions

Show that the following functions are convex:

$$\|x\|^2 = \sum_{i=1}^{p} x_i^2$$

$$\|x\|_1 = \sum_{i=1}^{p} |x_i|$$

$$F(x) = f(\langle x, y \rangle), \quad \text{for } y \in \mathbb{R}^p \text{ and } f : \mathbb{R} \to \mathbb{R} \text{ convex}$$

Smoothness

f is L-smooth $=$ ∇f is L-Lipschitz

!! f must be differentiable

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \forall x, y$$

Equivalent formulations:

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|x - y\|^2, \quad \forall x, y \in \mathbb{R}^p$$

$$\nabla^2 f(x) \preceq L \cdot I_p, \quad \forall x \in \mathbb{R}^p$$

!! If $\nabla^2 f$ is continuous, f is L-smooth on all compact sets

Strong convexity

f is μ - strongly convex if:

< 0 **so stronger than convexity !**

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)\frac{\mu}{2}\|x-y\|^2$$

Equivalent formulations:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle + \frac{\mu}{2}\|x-y\|^2, \quad \forall x, y \in \mathbb{R}^p$$

$$\nabla^2 f(x) \succeq \mu \cdot I_p, \quad \forall x \in \mathbb{R}^p$$